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Formation and propagation of a shock wave in a gas with temperature gradients

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A theoretical analysis was carried out to study the formation and propagation of a weak shock wave in a gas with longitudinal temperature gradients. An equation describing the formation and propagation of a weak shock wave through a non-uniform medium in the absence of energy dissipation was derived. An approximate analytical solution to the one-dimensional wave propagation equation is established. With this, the thermal gradient effects on the shock-wave Mach number and speed were investigated and the results were compared to earlier experiments. Numerical solutions for the same problem using Euler's equations have also been obtained and compared to the analytical results. The analysis shows that the time of shock-wave formation from the initial disturbance, for mild temperature gradients, is independent of the gradient. The shock wave forms at a longer axial distance from the initial disturbance when the temperature gradient is positive whereas the opposite is true for a negative temperature gradient.

1. Introduction

Earlier ballistic and shock-wave propagation experiments (Kimov *et al.* 1982; Basargin & Mishin 1984; Gorshkov *et al.* 1984; Klimov & Mishin 1990; Mishin, Serov & Yavor 1991; Bedin & Mishin 1995; Ganguly, Bletzinger & Garscadden 1998) indicate that shock waves passing through a zone of weakly ionized plasma may undergo substantial modification. In these experiments, the shock wave is observed to weaken and become diffuse. Although there has been some effort to explain these experimental observations, the fundamental mechanism(s) that cause these 'anomalies' reported in the experiments are not yet fully understood. The approaches that have been proposed can be divided into two classes. The first class of models relies on excess heating of the gas by the plasma and the above effects are explained by temperature gradients in the flow (thermal mechanism). In the second class of models, elementary processes between electrons and atoms/molecules in the plasma are brought into consideration (inherent plasma mechanism). The difficulty is, in part, due to the fact that the reported experiments were not carried out systematically and thermal gradients were always present. Therefore, the observed effects can be attributed to both plasma and thermal effects. In the present work, we attempt to shed light onto the problem by predicting the extent of longitudinal temperature gradient effects on the shock-wave modification observed in previous experiments.

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In an earlier work, one-dimensional shock-wave propagation through a non-uniform medium was investigated by Chisnell (1955). He treated the gradient region as a succession of uniform density subregions with discontinuous contacts at both ends and obtained an expression for the wave Mach number. Later, Sakurai (1960) sought a solution to the particular case where the gas density has a power dependence on the coordinate. An approximate analytical method for the calculation of the shock Mach number for a more general density dependence on the axial coordinate was developed by Whitham (1958; see also Whitham 1974). The approximate methods developed in these earlier reports met with a number of difficulties. For example, in the study of Sakurai (1960), the well-known effect of asymptotic relaxation of the shock wave was not found. In addition, the analysis of both Sakurai (1960) and Whitham (1958) led to the result that the shock-wave Mach number does not change after the wave leaves the non-uniform zone. On the other hand, numerical calculations based on the method of characteristics indicate that the wave Mach number does change over a certain distance downstream of the non-uniform zone (Bird 1961). It was also shown that for certain types of density distribution, the results obtained using the approximate methods of Chisnell (1955) and Whitham (1958) can deviate significantly from those obtained using the method of characteristics (Bird 1961). Naidis & Rumyantsev (1987) numerically calculated the shock-wave speed as it passed through a zone of higher gas temperature. The calculations were carried out for those conditions corresponding to the experiments of Basargin & Mishin (1984). The results of these calculations were in qualitative agreement with the experimental data. The studies discussed above deal only with the propagation of the shock wave; they do not allow for an analysis of the wave formation from an initial disturbance in presence of density (or temperature) gradients. More recently, Lin & Szeri (2001) carried out an analysis of the sonoluminescence process. Using the method of characteristics, they analysed the compression wave launched into a radially collapsing gas with a smoothly varying entropy ahead of the wave and established the time and location where the shock wave forms from the initial disturbance.

The objective of the present work is to develop an analytical solution to the problem of weak shock-wave formation and propagation in an ideal gas with a mild longitudinal temperature (or density) gradient. Thermal conductivity and viscosity are not taken into account. As pointed out by Whitham (1974), the modelling of shock-wave propagation without taking into account the energy dissipation describes adequately all features of the wave with the exception only of shock-wave thickness. From the basic conservation laws, we first obtain a one-dimensional wave equation in a gas with longitudinal non-uniformities. Next, we obtain an approximate solution to the governing equation, which is then used to study the temperature gradient effects on shock wave speed, Mach number and strength. The analytical results are compared to finite-difference-based numerical solutions of the Euler's equations as well as earlier glow discharge plasma experiments.

2. Disturbance propagation equation

In order to develop a one-dimensional propagation equation for a disturbance travelling through an ideal gas we use the following initial conditions: the pressure is constant throughout whereas the temperature and density distributions are non-uniform and the gas density is given by $\rho_0(x)$. We consider the propagation of weak waves into zones of mild temperature gradients in the absence of any heat sources. Under these conditions, the entropy is conserved on each particle path although the entropy of different particle paths may have different values. We also consider only the forward-propagating waves since the reflected wave through a zone of mild temperature is negligible.

We start with a one-dimensional disturbance of the gas density: $\rho(x,t) - \rho_0(x)$. The evolution of the disturbance is described by the continuity equation $\partial\rho(x,t)/\partial t + \partial Q(\rho,x,t)/\partial x = 0$, where $Q(\rho,x,t)$ is the mass flux. For a non-uniform medium, the continuity equation can be written as follows:

$$\frac{\partial\rho(x,t)}{\partial t} + v(x,t)\frac{\partial\rho(x,t)}{\partial x} + \frac{\partial Q(\rho,x,t)}{\partial x} = 0, \tag{2.1}$$

where, $v(x,t) = \partial Q(\rho,x,t)/\partial\rho$ is the disturbance propagation velocity.

Further, using the momentum and energy conservation laws and taking into consideration that entropy does not change on a given particle path, the following relations are valid:

$$v(x,t) = a(\rho,x) + u(x,t), \tag{2.2}$$

$$a(\rho,x) = a_0(x)\left[\frac{\rho(x)}{\rho_0(x)}\right]^{(\gamma-1)/2} + O\left(\frac{\Delta a_0}{a_0}\frac{u}{a_0}\right), \tag{2.3}$$

where $a(\rho,x)$ is the local speed of sound, $a_0(\rho,x)$ is the speed of sound in the undisturbed gas, Δa_0 is the change of the speed of sound over the length of the disturbance and u is the mass velocity of the flow and is defined as $u(x,t) = Q(x,t)/\rho(x,t)$. The second term on the right-hand side of (2.3) represents the order of magnitude of all other terms that are not included. Therefore, if the disturbance is weak ($u/a \ll 1$), and the change in the speed of sound is insignificant over the length of the disturbance ($\Delta a_0/a_0 \ll 1$), (2.3) will coincide with that for an isentropic flow (see, for example, Whitham 1974) since

$$\left(\frac{\Delta a_0}{a_0}\frac{u}{a_0}\right) \approx 0.$$

For a weak wave, the density disturbance is also small so that $(\rho - \rho_0)/\rho_0 \ll 1$ at any x. Using Taylor's expansion for a small density perturbation, (2.3) can be linearized to give

$$a(\rho,x) = a_0(x)\left[1 + \frac{\gamma-1}{2}\frac{\rho(x)-\rho_0(x)}{\rho_0(x)}\right] + O\left(\left(\frac{\rho-\rho_0}{\rho_0}\right)^2\right).$$

Using this along with $Q = \rho u$, we obtain the following for the disturbance propagation velocity and the mass flux:

$$v(x,t) = a_0(x)\left[1 + \frac{\gamma+1}{2}\frac{\rho(x)-\rho_0(x)}{\rho_0(x)}\right] + O\left(\left(\frac{\rho-\rho_0}{\rho_0}\right)^2\right), \tag{2.4a}$$

$$Q(\rho,x) = a_0(x,t)\left\{\rho(x) - \rho_0(x) + \frac{\gamma+1}{4}\left[\frac{\rho(x)-\rho_0(x)}{\rho_0(x)}\right]^2 + O\left(\left(\frac{\rho-\rho_0}{\rho_0}\right)^3\right)\right\}. \tag{2.4b}$$

From (2.3), we obtain for the density

$$\rho(x,t) = \rho_0(x)\left\{1 + \frac{2}{\gamma+1}\left[\frac{v(x,t)}{a_0(x)} - 1\right] + O\left(\left(\frac{v(x,t)-a_0}{a_0}\right)^2\right)\right\}. \tag{2.4c}$$

Further, calculating derivatives for each term in (2.1) and expressing them in terms of $a_0(x)$, $\rho_0(x)$ and $v(x,t)$, we obtain the following set of equations:

$$\frac{\partial \rho}{\partial t} = \frac{2}{\gamma+1}\frac{\rho_0(x)}{a_0(x)}\frac{\partial v}{\partial t}, \tag{2.5a}$$

$$\frac{\partial \rho}{\partial x} = \frac{2}{\gamma+1}\frac{\rho_0(x)}{a_0(x)}\frac{\partial v}{\partial x} - \frac{2}{\gamma+1}\frac{\rho_0(x)v}{a_0^2}\frac{\mathrm{d}a_0(x)}{\mathrm{d}x} + \frac{\mathrm{d}\rho_0(x)}{\mathrm{d}x}\left\{1+\frac{2}{\gamma+1}\left[\frac{v}{a_0(x)}-1\right]\right\}, \tag{2.5b}$$

$$\frac{\partial Q(\rho,x,t)}{\partial x} = \frac{\rho_0(x)}{\gamma+1}\frac{\mathrm{d}a_0(x)}{\mathrm{d}x}\left\{\left[\frac{v}{a_0(x)}-1\right]^2+2\left[\frac{v}{a_0(x)}-1\right]\right\}$$
$$-a_0(x)\frac{\mathrm{d}\rho_0(x)}{\mathrm{d}x}\left\{1+\left[\frac{v}{a_0(x)}-1\right]+\frac{1}{\gamma+1}\left[\frac{v}{a_0(x)}-1\right]^2\right\}. \tag{2.5c}$$

In system (2.5) we have neglected all terms of $O(((\rho-\rho_0)/\rho_0)^2)$ keeping just the linear terms in the expansions. Finally, substituting (2.5) into (2.1) the following equation is obtained for the propagation velocity of the disturbance:

$$\frac{\partial v(x,t)}{\partial t} + v(x,t)\frac{\partial v(x,t)}{\partial x} = v(x,t)\frac{\mathrm{d}a_0(x)}{\mathrm{d}x}. \tag{2.6}$$

This equation governs the propagation of a weak disturbance in a non-uniform medium with a mild temperature (or density) gradient. When $\mathrm{d}a_0(x)/\mathrm{d}x = 0$, (2.6) reduces to the well-known equation for a uniform medium (Whitham 1974).

3. Evolution of an initial disturbance and the formation of the shock wave

The shock wave occurs at a point in space where the first derivatives of the speed of sound and the flow rate become infinite. Until this instant, functions $v(x,t)$, $\rho(x,t)$ and their first derivatives are continuous and $v(x,t)$ satisfies (2.6) in the domain $x \in (-\infty,\infty)$. To obtain a solution to (2.6), we introduce a new dimensionless time $\tau = \omega t$ and coordinate $\xi = \kappa x$ where ω and κ, with dimensions s^{-1} and m^{-1}, respectively, are related to one another through $\omega/\kappa = a_0$ where $a_0 = \lim_{x\to-\infty} a_0(x)$. If the disturbance is limited in space, κ may be scaled with the disturbance width, X_L, such that $\kappa = 1/X_L$ and $\omega = a_0/X_L$. Then, for a new dimensionless disturbance velocity

$$\bar{v}(\tau,\xi) \equiv \frac{v\left(\dfrac{\tau}{\omega},\dfrac{\xi}{\kappa}\right)}{a_0} = \bar{a}_0(\xi) + \tfrac{1}{2}(\gamma+1)\frac{u\left(\dfrac{\tau}{\omega},\dfrac{\xi}{\kappa}\right)}{a_0},$$

where $\bar{a}_0(\xi) \equiv a_0(\xi/\kappa)/a_0$, (2.6) becomes:

$$\frac{\partial \bar{v}}{\partial \tau} + \bar{v}\frac{\partial \bar{v}}{\partial \xi} = \bar{v}\psi(\xi). \tag{3.1}$$

Here, $\psi(\xi) \equiv \partial\bar{a}_0(\xi)/\partial\xi$. Suppose that, at the instant $\tau = 0$, a disturbance occurs, so that

$$\bar{v}(\tau,\xi)|_{\tau=0} = \bar{a}_0(\xi) + f(\xi), \tag{3.2}$$

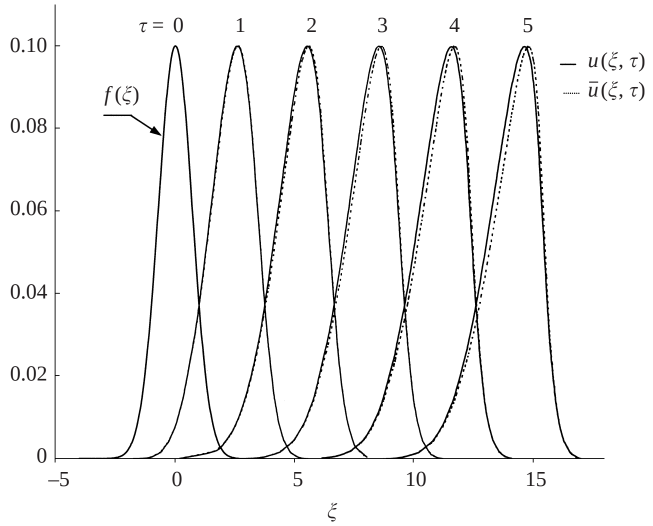

FIGURE 1. Comparison between ——, numerical solutions of (2.6) and - - -, the approximation given by (3.3). $f(\xi)$ is the initial distribution of the disturbance velocity.

where $f(\xi)$, observably, describes the initial shape of the disturbance. Equation (3.1) together with initial condition (3.2) describes the evolution of the initial disturbance. As shown in the Appendix A, when the initial disturbance is restricted in space, and the thermal gradient is small compared to the initial disturbance gradient, an analytical solution to (3.1) can be obtained. It is given by the following transcendental equation:

$$\bar{u}(\xi,\tau) = f(\varphi(\xi,\tau) - \bar{u}\tau), \tag{3.3}$$

where $\bar{u} \equiv \bar{v} - \bar{a}_0(\xi)$, and $\varphi(\xi,\tau)$ is expressed through $\bar{a}_0(\xi)$ and τ through

$$\tau = \int_{\varphi(\xi,\tau)}^{\xi} \frac{\mathrm{d}x}{\bar{a}_0(x)}.$$

Note that, if the temperature distribution in the medium is uniform ($\bar{a}_0(\xi) = 1$), then $\varphi(\xi,\tau) = \xi - \tau$, and $\bar{u} = f(\xi - \tau - \bar{u}\tau)$. A solution for this special case was obtained by Velikhovic & Lieberman (1987), for an initial disturbance shape of $f(\xi) \sim \sin\xi$. We compared the approximate analytical solution of (3.3) to numerical solutions of (2.6) for various distributions of $\bar{a}_0(\xi)$ and $f(\xi)$ with good agreement between the two. Figure 1 shows one such comparison. Here, the speed of sound in the thermal gradient region, and the initial disturbance are modelled by the functions $\bar{a}_0(\xi) = 2 + (2/\pi)\arctan\xi$ and $f(\xi) = u|_{\tau=0} = 0.1\exp(-\xi^2)$, respectively. As time progresses, the disturbance gradient becomes steeper for both solutions, tending towards a shock wave.

In order to find the time, τ_d, when the initial disturbance becomes a discontinuity forming the shock wave, the following system of equations have to be solved (Zel'dovich & Raizer 1967):

$$\left(\frac{\partial\xi}{\partial\bar{u}}\right)_\tau = 0, \quad \left(\frac{\partial^2\xi}{\partial\bar{u}^2}\right)_\tau = 0. \tag{3.4a,b}$$

Using (3.3), (3.4) can be written in the form:

$$1+\tau\frac{\partial f}{\partial g}=0, \quad \frac{\partial^2 f}{\partial g^2}=0, \tag{3.5a, b}$$

where $g=\varphi(\tau,\xi)-\bar{u}\tau$. It follows from (3.5$a$) that a necessary condition for discontinuity is the existence of such a location, ξ, where $\partial f/\partial\xi<0$. This condition implies that the wave is a compression wave and that a 'decompression' wave cannot exist. A solution to (3.5b) is a coordinate $g=g_m$ where the derivative $\partial f/\partial g$ is a maximum. The time of the shock-wave formation, τ_d, can then be calculated from

$$\tau_d=\frac{1}{|\mathrm{d}f/\mathrm{d}\xi|_{max}}. \tag{3.6a}$$

The location where the discontinuity occurs, ξ_d, is given by:

$$\xi_m=\varphi(\tau_d,\xi_d)-f(\xi_m)\tau_d. \tag{3.6b}$$

Here, ξ_m is the location where $\mathrm{d}f/\mathrm{d}\xi$ is a maximum. This equation is the same as that for a disturbance travelling through a uniform medium, which was given by Whitham (1974). Next, we calculate the speed of propagation for the initial disturbance up to the point where the discontinuity is formed. We assume that the initial disturbance function, $f(\xi)$, has a single maximum f_{max} at $\xi=0$. For a uniform medium, the solution to (3.5) is $\xi(\tau)-\tau-f_{max}\tau=0$, where $\xi(\tau)$ is the position of the wave maximum at time τ, and for the speed of the disturbance maximum, we obtain $V_d=\mathrm{d}\xi/\mathrm{d}\tau=1+f_{max}$. Similarly, the speed of the disturbance maximum for a non-uniform region can be found from the following system of equations:

$$\varphi(\xi(\tau),\tau)-f_{max}\tau=0, \quad \int_{\varphi(\xi(\tau),\tau)}^{\xi(\tau)}\frac{\mathrm{d}x}{\bar{a}_0(x)}=\tau, \quad V_d=\frac{\mathrm{d}\xi(\tau)}{\mathrm{d}\tau}, \tag{3.7a–c}$$

which has the solution

$$V_d=\bar{a}_0(\xi)+f_{max}\frac{\bar{a}_0(\xi(\tau))}{\bar{a}_0(f_{max}\tau)}. \tag{3.8}$$

Here, $\xi(\tau)$ is a solution of (3.7b) at $\varphi(\xi(\tau),\tau)=f_{max}\tau$.

Equation (3.6) shows that the time of shock-wave formation is independent of the temperature distribution in the medium and, for the weak shock wave, it is defined solely by the initial disturbance. On the other hand, the position where the shock wave forms depends on $\bar{a}_0(\xi)$. If the temperature gradient is positive, the speed of the disturbance maximum V_d increases (compared to that for uniform temperature) and hence it takes a longer distance to form the shock wave from the initial disturbance. For a negative temperature gradient, the opposite is true.

4. Propagation of a weak shock wave through a non-uniform medium

Here, we consider the propagation of the disturbance for $\tau>\tau_d$. As mentioned earlier, for $\tau\geqslant\tau_d$, the distribution of the bulk velocity $u(\xi,\tau)$ and its first derivative are not continuous over their arguments. Therefore, (2.6) is not valid in the entire domain $\xi\in[-\infty,\infty]$. However, both $u(\xi,\tau)$ and its derivatives are continuous separately in the half-domains $\xi>\xi_d(\tau)$ and $\xi<\xi_d(\tau)$ where $\xi_d(\tau)$ is the position of the discontinuity. Therefore, for an adequate description of the shock wave, (2.6) must be solved separately for each of the half-domains and the solutions linked together at the discontinuity point. From (3.3), it follows that

$$\left.\begin{aligned} \bar{u}_1 &= f(\varphi(\xi,\tau) - \bar{u}_1\tau), \quad \int_{\varphi(\xi,\tau)}^{\xi} \frac{\mathrm{d}x}{\bar{a}_0(x)} = \tau \quad (\xi < \xi_d(\tau)), \\ \bar{u}_2 &= f(\varphi(\xi,\tau) - \bar{u}_2\tau), \quad \int_{\varphi(\xi,\tau)}^{\xi} \frac{\mathrm{d}x}{\bar{a}_0(x)} = \tau \quad (\xi > \xi_d(\tau)). \end{aligned}\right\} \tag{4.1a}$$

At the discontinuity point, $\xi = \xi_d(\tau)$, these are:

$$\bar{u}_{1d} = f(z_1), \quad \bar{u}_{2d} = f(z_2), \quad \int_{z_1+f(z_1)\tau}^{\xi_d(\tau)} \frac{\mathrm{d}x}{\bar{a}_0(x)} = \tau, \quad \int_{z_2+f(z_2)\tau}^{\xi_d(\tau)} \frac{\mathrm{d}x}{\bar{a}_0(x)} = \tau, \tag{4.1b}$$

where $z_1 = \varphi(\xi_d(\tau),\tau) - \bar{u}_{1d}\tau$, $z_2 = \varphi(\xi_d(\tau),\tau) - \bar{u}_{2d}\tau$ $(z_1 > z_2)$. The bulk velocity increases through the discontinuity point: $\bar{u}_{1d} < \bar{u}_{2d}$, because $\mathrm{d}f(\xi)/\mathrm{d}\xi|_{\xi=\xi_d} < 0$ and $\tau = -(z_1 - z_2)/(f(z_1) - f(z_2))$. The equations for the position of discontinuity could be obtained using conservation laws. If function $\bar{a}_0(\xi)$ and its derivative are continuous, the velocity of the discontinuity plane is given by

$$\dot{\xi}_d(\tau) = \bar{a}_0(\xi) + \tfrac{1}{2}[f(z_1) + f(z_2)], \tag{4.2}$$

where the overdot denotes the derivative over time, τ. Equation (4.2), together with (4.1*b*) forms a system of three nonlinear integro-differential equations for functions $z_1(\tau)$, $z_2(\tau)$ and $\xi_d(\tau)$. The solution of (4.1*b*) and (4.2) fully describes the propagation of a weak shock wave in the non-uniform region. In differential form, the equations can be written as follows:

$$\left.\begin{aligned} \dot{\xi}_d(\tau) &= \bar{a}_0(\xi_d) + \tfrac{1}{2}(\dot{z}_1 + \dot{z}_2)[1 + \tau(f'(z_1) + f'(z_2))]\frac{\bar{a}_0(\xi_d)}{\bar{a}_0\left[\frac{1}{2}(z_1 + z_2 + \tau(f(z_1) + f(z_2)))\right]}, \\ \dot{\xi}_d(\tau) &= \bar{a}_0(\xi_d) + \tfrac{1}{2}[f(z_1) + f(z_2)], \\ \tau &= \frac{z_1 - z_2}{f(z_2) - f(z_1)}. \end{aligned}\right\} \tag{4.3a}$$

The prime indicates a derivative over each function's respective argument. System (4.3*a*) should satisfy initial conditions:

$$\begin{gathered} \xi_d(\tau)|_{\tau=\tau_d} = \xi_d, \quad \dot{\xi}_d(\tau)|_{\tau=\tau_d} = \bar{a}_0(\xi_d) + f(\xi_m)\frac{\bar{a}_0(\xi_d)}{\bar{a}_0(f(\xi_m)\tau)}, \\ z_1(\tau)|_{\tau=\tau_d} = z_2(\tau)|_{\tau=\tau_d} = \xi_m \end{gathered} \tag{4.3b}$$

where, ξ_d is determined from (3.6*b*), and ξ_m is the coordinate where $|f'(\xi)|$ is a maximum.

It was not possible to find an analytical solution to (4.3*a*) and (4.3*b*) for arbitrary $f(\xi)$ and $\bar{a}_0(\xi)$. Instead, we carried out an analysis of the asymptotic behaviour of the discontinuity at $\tau \to \infty$. Consider a case where function $f(\xi)$ is non-zero only in the range $(0, \xi_L)$ and it has a single maximum inside this range. An asymptotic solution for the shock-wave velocity and the wave Mach number in such a case was obtained as follows:

$$\frac{M(\bar{\xi}) - 1}{M_0 - 1} = \frac{\bar{w}(\bar{\xi})}{\bar{a}_0(\bar{\xi})\left[1 + \displaystyle\int_1^{\bar{\xi}} \frac{\mathrm{d}x}{\bar{a}_0(x)}\right]}, \tag{4.4}$$

$$V(\bar{\xi}) = a_0(1)\left\{\bar{a}_0(\bar{\xi})+(M_0-1)\exp\left\{\frac{1}{2}\int_1^{\bar{\xi}}\frac{\mathrm{d}y}{\bar{a}_0^2(y)\left[1+\displaystyle\int_1^y\frac{\mathrm{d}x}{\bar{a}_0(x)}\right]}\right\}\bigg/\left[1+\int_1^{\bar{\xi}}\frac{\mathrm{d}x}{\bar{a}_0(x)}\right]\right\}. \tag{4.5}$$

Here,

$$\bar{w}(\bar{\xi}) = \exp\left\{\frac{1}{2}\int_1^{\bar{\xi}}\frac{\mathrm{d}y}{\bar{a}_0^2(y)\left[1+\displaystyle\int_1^y\frac{\mathrm{d}x}{\bar{a}_0(x)}\right]}\right\},$$

$\bar{\xi} = \xi_d/\xi_{d_0}$ (ξ_{d_0} is the initial position of the discontinuity), M_0 and $a_0(1)$ are the initial values of the shock-wave Mach number and the speed of sound (upstream of the thermal gradient zone), in dimensional form. The details of the asymptotic solution are presented in Appendix B.

5. Discussion

Equations (4.4) and (4.5) give the Mach number and velocity, respectively, for weak shock waves travelling into zones of mild temperature (or density) gradients. Figure 2 shows the Mach-number evolution of shock waves travelling through a negative temperature gradient zone. The temperature gradient is confined in the region $0 < \xi < 1$. For $\xi > 1$, the temperature is uniform. The local Mach numbers calculated using (4.4) are compared to the earlier approximate method of Chisnell (1955) and Whitham (1958) who treated the gradient zone as a region of small piecewise temperature changes. The present results are also compared to those obtained earlier by Bird (1961) who used the method of characteristics for the same thermal gradient conditions and initial wave Mach number. In order to provide a further assessment of the present analytical results, we obtained computational solutions to the same one-dimensional shock-wave propagation problem by solving Euler's equations numerically using a second-order accurate finite-difference technique. The solutions to the unsteady inviscid one-dimensional conservation equations (mass, momentum and energy) as well as the equation of state were obtained using the MacCormack scheme (MacCormack 1969). The scheme features a two-step predictor–corrector discretization of the governing equations. A forward difference is used in the predictor step for the convective derivative, whereas a backward difference is used in the corrector step. The scheme is second-order accurate and conditionally stable. The analytical solutions compare well with the present numerical calculations for all the initial Mach number cases studied. The results of Bird, Chisnell and Whitham for the mildest shock-wave case of $M_0 = 1.1$ also agree well with the present calculations. On the other hand, for $M_0 = 2$ the Chisnell–Whitham method significantly overestimates the wave Mach number through the gradient region whereas Bird's calculation is still in reasonable agreement with the present calculations.

For positive temperature gradients (when the speed of sound increases – or the gas density decreases – in the direction of the shock wave propagation), the Mach number decreases and the shock-wave velocity increases with axial distance. The opposite is also true: a negative temperature gradient leads to increasing shock Mach numbers and decreasing shock velocities along the propagation direction. These trends are in agreement with the experimental results of Piskareva & Shugaev (1978) who observed increasing shock velocities through a decreasing temperature gradient

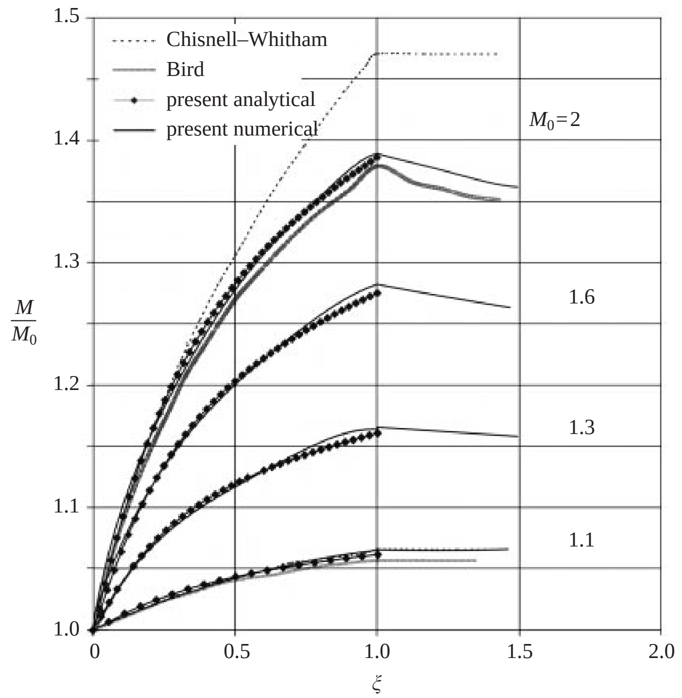

FIGURE 2. Evolution of the shock-wave Mach number travelling through a negative temperature gradient zone ($0 < \xi < 1$) with $T/T_0 = (1 + 7\xi)^{-1}$. $T_0 = T(0)$.

region. Figure 3 illustrates this point. In the figure, the change in the shock-wave Mach number through a parabolic temperature distribution is demonstrated. The initial and final temperatures at the two ends of these gradient regions are kept the same. The normalized Mach number $(M(\bar{\xi}) - 1)/(M_0 - 1)$ calculated from (4.4), is plotted against the normalized distance, $\bar{\xi} = \xi_d/\xi_{d_0}$, where ξ_{d_0} is the initial position of the shock wave. In the same figure, the present numerical inviscid calculations are also presented which agree reasonably well with those obtained using (4.4). The Mach numbers calculated using the approximate formula $(M(\bar{\xi}) - 1)/(M_0 - 1) = (\rho(\bar{\xi})/(\rho(1))^{1/4}$ are also presented in the figure. This simple formula was proposed by Naidis & Rumyantsev (1987) and is based on the approximate solution obtained earlier by Whitham (1974). All three sets of calculations show similar trends for the Mach number within the gradient zone. However, the approximation of Naidis & Rumyantsev (1987) underestimates the peak Mach number. This may be because the formula was obtained without taking into account the changes in the wave structure while travelling through the gradient zone. The fact that there is an asymptotic decay of the shock wave in a uniform medium even in the absence of dissipation is well known. In figure 3, this discrepancy is clear in the uniform zone downstream of the gradient region: the present calculations, both analytical and numerical, show a mild decay in the wave Mach number past the gradient region.

To the best of our knowledge, the propagation of weak shock waves through non-uniform media has not been studied experimentally. Therefore, we compare our calculations with the experimental data of Piskareva & Shugaev (1978) who studied the propagation of a shock wave with $M_0 = 1.9$. This Mach number is somewhat large for the wave to be considered ‘weak’. However, the wave velocity calculated using (4.5) compares reasonably well with the experimental data, as shown in figure 4.

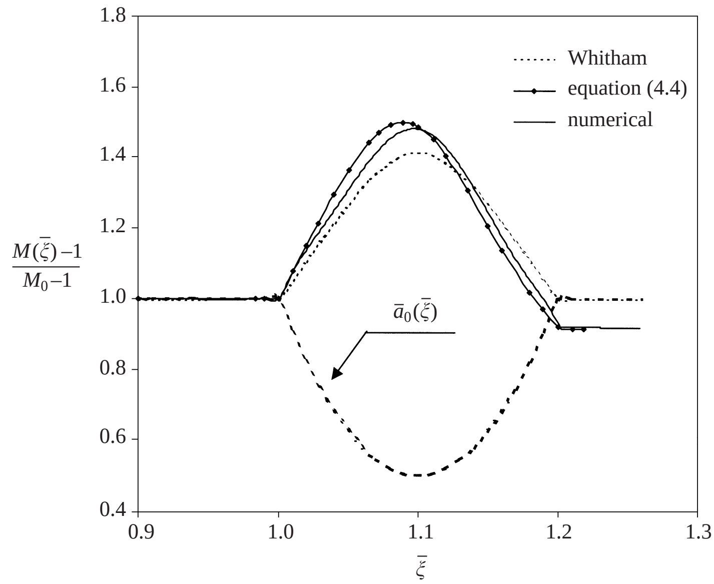

FIGURE 3. Mach number evolution for a shock wave travelling through a region of negative parabolic temperature distribution. (The acoustic speed distribution in the medium is given by the dashed line.)

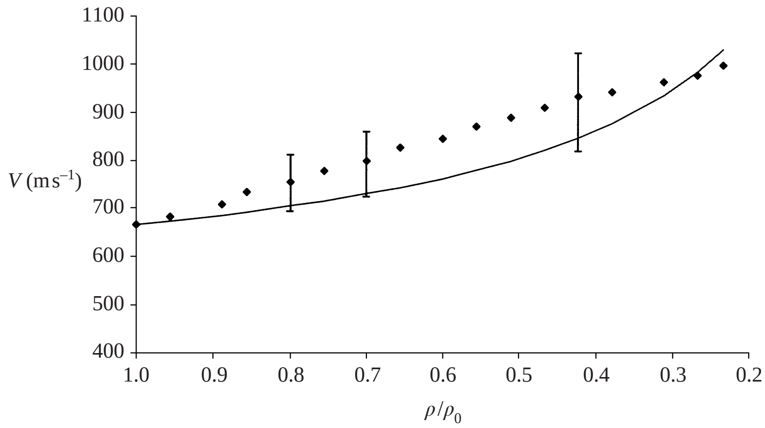

FIGURE 4. Evolution of shock-wave velocity through a gradient region. ——, calculation using (4.5). ◆, measurements by Piskareva & Shugaev (1978).

Here, we use the same negative and linear density gradient reported by Piskareva & Shugaev for their experiments. The error bars in the measured data are those reported by Piskareva & Shugaev. In the figure, the velocity is presented against the local density normalized by the density upstream of the gradient region. Again, despite the fact that the shock wave for this experiment (with $M_0 = 1.9$) cannot be considered weak ($M_0 - 1 \ll 1$), the agreement between the experiments and the calculations is still good.

In order to further assess the present analysis, we use the experimental results of Klimov *et al.* (1982) and Basargin & Mishin (1984) who studied the propagation of shock waves in glow discharge plasmas. Although the main purpose of these experiments was to study the effect of the glow discharge plasma on shock-wave

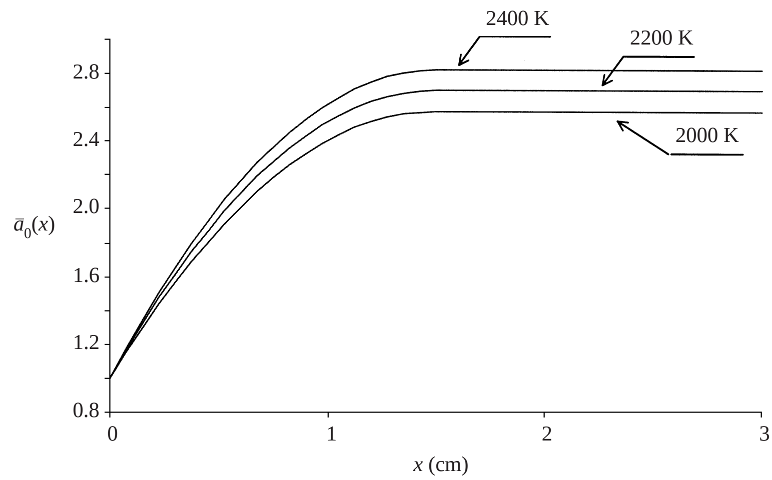

FIGURE 5. Estimated acoustic speed distributions for the K and BM experiments for maximum centreline temperatures of 2000, 2200 and 2400 K.

characteristics, significant thermal gradients existed in the positive column of the plasma region which, undoubtedly, influenced their results. The thermal gradients are caused by the plasma heating and thermal losses to the environment. In these experiments, changes occurred in the speed and structure of the wave as it travelled through the plasma region. The authors attributed these changes to an unspecified inherent plasma mechanism alone, without giving consideration to longitudinal thermal gradient effects. The present calculations can be used to re-analyse these experiments and help assess plasma effects by determining the extent of the thermal effects. In order to provide quantitative evaluation of the thermal effects in the Klimov *et al.* and Basargin & Mishin (hereinafter referred to as K and BM, respectively) experiments using our analysis, we must establish a reliable estimate of the centreline temperature in those experiments. The authors of these experimental studies conjectured that the centreline temperature in the plasma did not exceed 1000 K at the plasma current density of 30 mA cm^{-2}. The discharges in K and BM experiments were formed in cylindrical tubes of diameters ranging between 1 and 6 cm that were filled with air at d.c. current densities of 30–40 mA cm^{-2}. Based on the experimental results of Golubovsky & Telezhko (1983, 1984) who measured centreline temperatures of nitrogen and oxygen mixture plasmas under similar conditions to those in K and BM experiments, and on the analysis presented by Naidis & Rumyantsev (1987), we conclude that the centreline temperature in the K and BM experiments should be in the range of 2200–2400 K. The discrepancy may be explained by the fact that the gas temperature in the K and BM experiments was inferred from a laser interferometer with the probe beam sent across the discharge tube. This is a line-of-sight measurement technique that provides the gas density averaged over the length of test section. Therefore, the inferred temperature is spatially averaged and it is likely to be significantly lower than the centreline value. In contrast, Golubovsky & Telezhko (1983, 1984) were able to measure the radial distribution of gas temperature using two independent experimental techniques. Using the axial temperature gradient distribution reported in BM experiments (which would be much less influenced by the measurement technique used) and fitting this data to a fourth-order polynomial, we calculated the variation of the acoustic speed for several downstream centreline temperatures around the value of 2200 K. Figure 5 shows three of the speed of sound

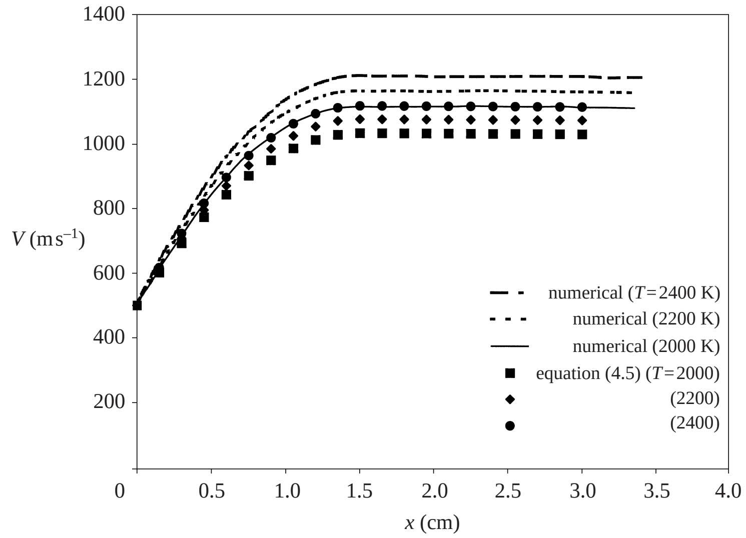

FIGURE 6. Shock-wave velocities calculated for acoustic speed gradients given in figure 5.

distributions which are normalized by the upstream condition whereas figure 6 shows the calculated shock-wave velocities corresponding to the three acoustic speed distributions given in figure 5. The initial velocity at the upstream edge of the gradient region was taken to be 500 ms^{-1} which is the value reported by K and BM. A positive temperature gradient along the shock-wave propagation direction leads to an increase in the shock-wave velocity and the terminal velocity (at the end of the gradient region) corresponds to the highest downstream centreline temperature. Although there is excellent qualitative agreement between the numerical and the analytical solutions, the terminal velocities obtained by the numerical solution are about 7% higher than those obtained from (4.5) for each temperature case. On average, the terminal velocity of 1100 ms^{-1} reported by K and BM agrees well with the present results. Figure 7 illustrates the dependence of the terminal shock-wave velocity on the initial wave velocity at the upstream edge of the thermal gradient region. It was calculated using (4.5) and for a centreline temperature of 2400 K. A relatively weak and nearly linear dependence of terminal shock velocity on the initial velocity is observed. A change of a factor of two in the initial velocity results in a change of the velocity at the end of the gradient region of only about 25%. This finding is also consistent with the experimental results of K and BM who reported a change of 10% in the shock-wave velocity in plasma when the initial velocity was changed by 60%.

Another observation made by K and BM, as well as in other experiments studying shock-wave behaviour in weakly ionized plasmas, was the broadening and the perceived weakening of the shock-wave. Axial thermal gradients may indeed be at least partially responsible for this effect as well. In figure 8, shock-wave Mach numbers calculated using (4.4) are presented for the acoustic speed distribution shown in figure 5 (for $T = 2400$ K case). The shock-wave Mach number decreases as it travels through the gradient region (to about 1.17 from 1.47). This significant decrease in the Mach number should lead to a weakening of the wave. Therefore, the weakening of the shock wave observed in K and BM, as well as in some of the other plasma interaction experiments, may be caused, at least partially, by the axial temperature gradients.

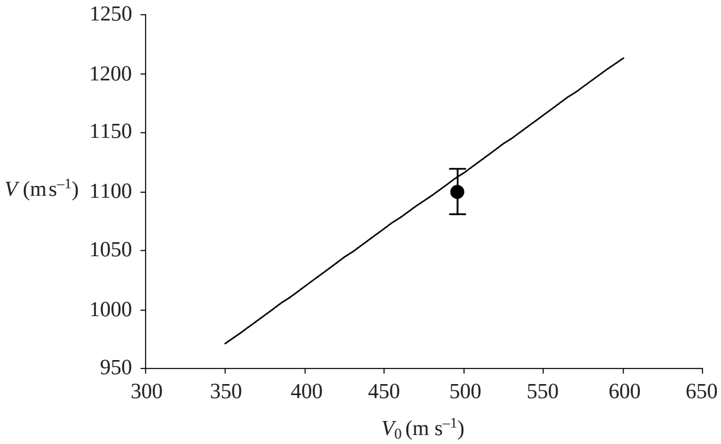

FIGURE 7. Dependence of the terminal shock-wave velocity on the velocity upstream of the gradient region. ——, calculations using (4.5). ●, measurement by Klimov *et al.* (1982).

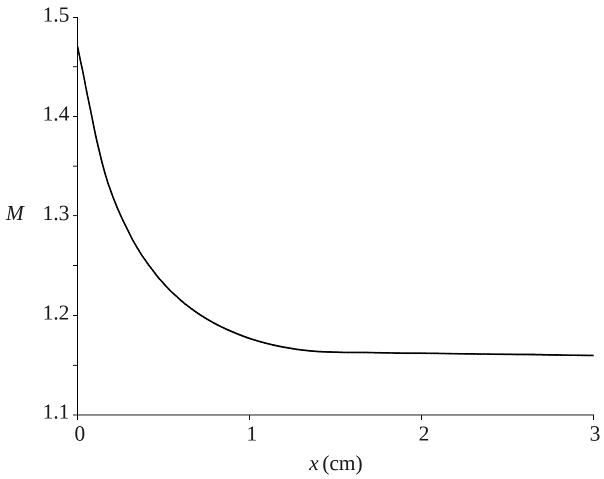

FIGURE 8. Calculated shock-wave Mach numbers in the gradient region using (4.4). The acoustic speed distribution corresponds to the 2400 K case in figure 5.

The influence of the temperature gradient on shock-wave broadening cannot be quantified by any model that neglects the energy dissipation processes. As discussed earlier in the analysis, if the solution for the wave propagation (2.6) becomes discontinuous at a certain time $\tau = \tau_d$, it remains discontinuous at all $\tau > \tau_d$ irrespective of the type of temperature gradient, $\bar{a}_0(\xi)$. In other words, the shock-wave width is always zero when thermal conductivity and viscosity effects are neglected in the model. However, a qualitative assessment of the temperature gradient effects on shock-wave thickness and strength can be made within the framework of our analysis. As shown in figure 8, the Mach number decreases when the wave is passing through a zone of positive temperature (or a negative density) gradient. The shock-wave intensity based on the density ratio, ρ_2/ρ_1, and the broadening of the shock wave, $\Delta(x)/\Delta_0$, may be evaluated using normal shock relations. (The subscripts 1 and 2 refer to the locations upstream and downstream of the shock wave, respectively; Δ is the shock wave thickness and Δ_0 is this thickness upstream of the gradient region.) The normal

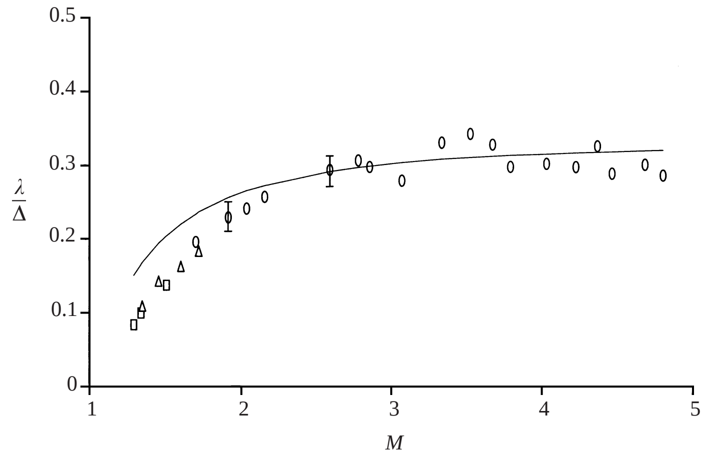

FIGURE 9. Variation of normalized inverse shock-wave thickness with Mach number. ——, $3\lambda[(P_2/P_1)/((P_2/P_1)-1)]$; ○, Linzer & Hornig (1963); □, Hansen & Hornig (1960); △, Talbot & Sherman (1959).

shock relations for density ratios are given by

$$\frac{\rho_2}{\rho_1}(M(x)) = \frac{M(x)^2(\gamma+1)}{2+(\gamma-1)M(x)^2},$$

$$\frac{\rho_2}{\rho_1}(M_0) = \frac{M_0^2(\gamma+1)}{2+(\gamma-1)M_0^2}.$$

The former equation is for the gradient region whereas the latter is for the uniform region upstream of the gradient zone. As for the thickness of the shock wave, Zel'dovich & Raizer (1967) proposed a general expression $\Delta \approx \lambda[M/(M^2-1)]$ where λ is the mean free path of the gas. For a weak shock wave, they proposed the further simplified expression $\Delta \approx \lambda\lfloor P_1/(P_2-P_1)\rfloor$ where P is the gas pressure. However, both expressions lead to $\Delta = 0$ for $M \to \infty$ and thus are not suitable as they estimate a shock-wave thickness smaller than λ at high Mach numbers. Here, we propose

$$\Delta = C\lambda\frac{P_2/P_1}{(P_2/P_1)-1}, \tag{5.1}$$

where, C is a constant. This equation is based on the shock strength and provides a set of more appropriate asymptotes. Further, it approximates the shock thickness fairly well in the moderate Mach number range, as shown in figure 9. In the figure, (5.1) is compared to the earlier experimental results of Talbot & Sherman (1959), Hansen & Hornig (1960) and Linzer & Hornig (1963). The solid line in the figure is a fit to the experimental data and represents the expression $3\lambda[(P_2/P_1)/((P_2/P_1)-1)]$. Using (5.1), we now calculate the relative thickness of the shock wave as it travels through the gradient region in the experiments of K and BM. Figure 10 shows the relative thickness of the shock wave along with the relative change in the density ratio. Here, the pressure ratios are obtained using normal shock relations and $M(x)$ is calculated using (4.4). The specific heat ratio is taken to be $\gamma = 1.4$. An initial velocity $V_0 = 500\,\mathrm{m\,s^{-1}}$ was used in the calculations and the centreline maximum temperature was taken to be 2400 K. At the end of the temperature gradient region, the shock wave becomes significantly weaker with a density ratio of only about 70% of that at the upstream edge of the gradient region. The thickness of the shock wave is also significantly altered; at the end of the region, it is nearly doubled.

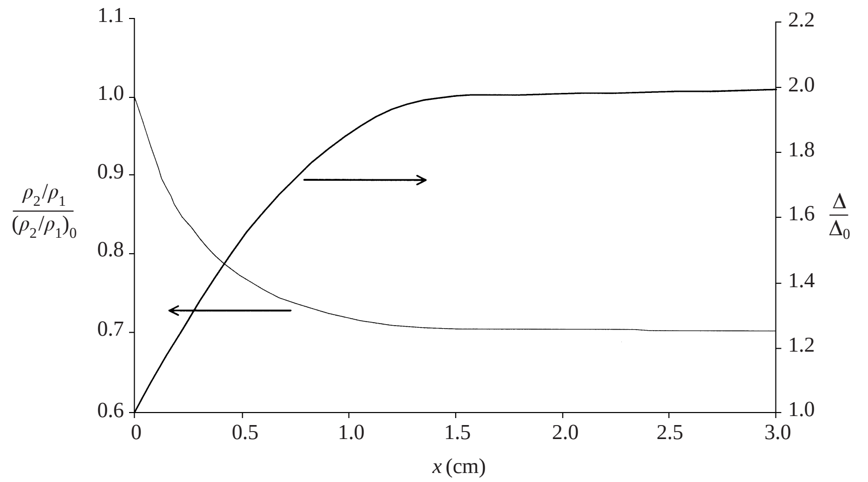

FIGURE 10. The evolution of the shock-wave density ratio and thickness with axial distance in the gradient region.

6. Concluding remarks

The present analysis shows that axial temperature gradients may significantly alter the propagation of weak shock waves. In most experimental studies of shock wave propagation through glow discharge plasmas, axial temperature gradients did exist at least at the upstream and downstream boundaries of the plasma. Therefore, thermal gradients must have contributed to some of the observed anomalous effects of shock-wave propagation through glow discharge plasmas such as the increase in shock-wave velocity and thickness and decrease in shock strength. Figure 10 clearly shows this significant thermal effect which is in good qualitative agreement with experimental observations of Klimov *et al.* (1982) and Basargin & Mishin (1984).

Appendix A. Solution for equation (3.1)

We seek a solution for (3.1) with initial condition (3.2) in the form

$$\bar{v} = \bar{a}_0(\xi) + f(\xi + \Phi(\xi,\tau) - \bar{v}\tau), \tag{A 1}$$

where $\Phi(\xi,\tau)$ is the unknown function. Substituting (A 1) into (3.1) and (3.2) we obtain

$$\frac{\partial \Phi}{\partial \tau} + \bar{v}\frac{\partial \Phi}{\partial \xi} = \tau\bar{v}\Psi(\xi), \quad \Phi(\xi,\tau)|_{\tau=0} = 0.$$

Further, replacing $\Phi(\xi,\tau)$ with function $F(\xi,\tau) = \Phi(\xi,\tau) - \bar{a}_0(\xi)\tau$, (3.1) and (3.2) become

$$\frac{\partial F}{\partial \tau} + \bar{v}\frac{\partial F}{\partial \xi} = -\bar{a}_0(\xi), \quad F(\xi,\tau)|_{\tau=0} = 0. \tag{A 2}$$

We will be seeking a solution to this equation by iterations. Let $\bar{u} \equiv \bar{v} - \bar{a}_0(\xi)$. Then, the connection between $(n-1)$th and nth iterations is given by

$$\frac{\partial F_n}{\partial \tau} + \bar{a}_0(\xi)\frac{\partial F_n}{\partial \xi} = -\bar{a}_0(\xi) - \bar{u}_{n-1}(\xi,\tau)\frac{\partial F_{n-1}}{\partial \xi}, \quad F_n(\xi,\tau)|_{\tau=0} = 0, \tag{A 3}$$

where $\bar{u}_{n-1}$ is defined as $\bar{u}_{n-1} = f(\xi + F_{n-1}(\xi,\tau) - \bar{u}_{n-1}\tau)$. A zero-order solution $F_0(\xi,\tau)$, found from

$$\frac{\partial F_0}{\partial \tau} + \bar{a}_0(\xi)\frac{\partial F_0}{\partial \xi} = -\bar{a}_0(\xi), \quad F_0(\xi,\tau)|_{\tau=0} = 0, \tag{A 4}$$

is

$$F_0(\xi,\tau) = -\int_0^{\tau} \bar{a}_0[\varphi(\xi,\tau,\tau')]\,\mathrm{d}\tau'. \tag{A 5}$$

Here, $\varphi(\xi,\tau,\tau')$ is a solution of the transcendental equation

$$\tau - \tau' = \int_{\varphi(\xi,\tau,\tau')}^{\xi} \frac{\mathrm{d}x}{\bar{a}_0(x)}. \tag{A 6}$$

Rewriting (A 5) for a new variable $y = \varphi(x,\tau,\tau')$ and calculating $\mathrm{d}\tau'/\mathrm{d}y$ from (A 6) we obtain

$$F_0 = \varphi(\xi,\tau) - \xi, \tag{A 7}$$

where $\varphi(\xi,\tau) = \varphi(\xi,\tau,\tau')|_{\tau'=0}$. Thus, zero-order approximate solution $\bar{u}_0$ is given by the following equation:

$$\bar{u}_0 = f(\varphi(\xi,\tau) - \bar{u}_0\tau) = f(\varphi(\xi,\tau) - \bar{u}_0\tau). \tag{A 8}$$

For the first-order approximation, $F_1(\xi,\tau)$, using (A 3), (A 7) and (A 8) we obtain:

$$F_1(\xi,\tau) = F_0(\xi,\tau) - \int_{\varphi(\xi,\tau)}^{\xi} u_0[x,\tau'(x,\tau,\xi)]\frac{1}{\bar{a}_0(x)}\left[1 - \frac{\bar{a}_0(x)}{\bar{a}_0(\varphi(\xi,\tau))}\right]\mathrm{d}x, \tag{A 9}$$

where

$$\tau'(x,\tau,\xi) = \tau + \int_{\xi}^{x} \frac{\mathrm{d}y}{\bar{a}_0(y)}.$$

Now, we evaluate the difference between the first- and zero-order approximations $|F_1(\xi,\tau) - F_0(\xi,\tau)|$. Suppose that function $f(\xi)$ is positive in its absolute maximum at $\xi = \xi_m : f(\xi_m) = f_{max} > 0$. Also, we suppose that the absolute value of the first derivative has a single maximum at $\xi = \xi'_m : |\mathrm{d}f(\xi)/\mathrm{d}\xi\|_{\xi=\xi'_m} = |\mathrm{d}f/\mathrm{d}\xi\|_{max}$ and the value of this maximum is greater than the maximum of the disturbance function itself: $f_{max}/|\mathrm{d}f/\mathrm{d}\xi\|_{max} \leqslant 1$. The last condition simply means that the initial disturbance is sharp enough. It is easy to see that it is well satisfied in a typical experiment. For example, for a harmonic acoustic wave of any amplitude, $f_{max}/|\mathrm{d}f/\mathrm{d}\xi\|_{max} = 1$. Suppose that $f(\xi)$ has a single 'hump' of width $\Delta\xi$. Then, $\Delta\xi \sim f_{max}/|\mathrm{d}f/\mathrm{d}\xi\|_{max}$. The integral in (A 9) can be evaluated as $\int \leqslant (f^2_{max}/\bar{a}^2_0(\xi))\Delta\xi(\Delta a_0)_{max}$. Using

$$(\Delta a_0)_{max} \leqslant (\mathrm{d}a_0/\mathrm{d}\xi)_{max}\Delta\xi,$$

we obtain

$$\int \leqslant \frac{\dfrac{f^2_{max}}{a^2_{min}}\left(\dfrac{\mathrm{d}a_0}{\mathrm{d}\xi}\right)_{max}}{\left(\dfrac{\mathrm{d}f}{\mathrm{d}\xi}\right)_{max}} \frac{f_{max}}{\left(\dfrac{\mathrm{d}f}{\mathrm{d}\xi}\right)_{max}},$$

and therefore for the difference $|F_1(\xi,\tau)-F_0(\xi,\tau)|$ we obtain

$$|F_1(\xi,\tau)-F_0(\xi,\tau)| \leqslant \frac{f_{max}^2}{\bar{a}_{0min}^2}\frac{\left|\dfrac{\mathrm{d}\bar{a}_0(\xi)}{\mathrm{d}\xi}\right|_{max}}{\left|\dfrac{\mathrm{d}f}{\mathrm{d}\xi}\right|_{max}}\frac{f_{max}}{\left|\dfrac{\mathrm{d}f}{\mathrm{d}\xi}\right|_{max}}, \tag{A 10}$$

where $\bar{a}_{0min}$ is the minimal value of $\bar{a}_0(\xi)$ in a range of distance between ξ and $\xi-\Delta\xi$, and $|\mathrm{d}\bar{a}_0(\xi)/\mathrm{d}\xi|_{max}$ is the maximum of the derivative in the same range. In studies of shock-wave propagation, weak and moderate shock waves are formed by an electrical discharge or by blowing a diaphragm. The thermal gradient is formed by heating the gas through the walls. In such conditions, the temperature does not change significantly over the length of the initial disturbance, i.e. $|\mathrm{d}\bar{a}_0(\xi)/\mathrm{d}\xi|_{max}/|\mathrm{d}f/\mathrm{d}\xi|_{max} \ll 1$. Therefore,

$$|F_{n-1}(\xi,\tau)-F_n(\xi,\tau)| \sim \frac{\left|\dfrac{\mathrm{d}a_0}{\mathrm{d}\xi}\right|_{max}}{\left|\dfrac{\mathrm{d}f}{\mathrm{d}\xi}\right|_{max}}\left(\frac{f_{max}}{\bar{a}_{0min}}\right)^{n+1} \ll 1.$$

For the overall difference between the exact solution and the zero-order approximation we then obtain:

$$\begin{aligned} F(\xi,\tau)-F_0(\xi,\tau) &= \lim\{F_0+(F_1-F_0)+(F_2-F_1)+\cdots+(F_n-F_{n-1})\}-F_0 \\ &= \sum_{k=0}^{\infty}(F_{k+1}-F_k) \leqslant \frac{\left|\dfrac{\mathrm{d}a_0}{\mathrm{d}\xi}\right|_{max}}{\left|\dfrac{\mathrm{d}f}{\mathrm{d}\xi}\right|_{max}}\sum_{k=0}^{\infty}\left(\frac{f_{max}}{\bar{a}_{0min}}\right)^{k+2} \\ &= \frac{\left|\dfrac{\mathrm{d}a_0}{\mathrm{d}\xi}\right|_{max}}{\left|\dfrac{\mathrm{d}f}{\mathrm{d}\xi}\right|_{max}}\frac{\left(\dfrac{f_{max}}{\bar{a}_{0min}}\right)^2}{\left(1-\dfrac{f_{max}}{\bar{a}_{0min}}\right)} \end{aligned}$$

and for the relative uncertainty

$$\frac{F(\xi,\tau)-F_0(\xi,\tau)}{F(\xi,\tau)} \ll \frac{\left|\dfrac{\mathrm{d}\bar{a}_0}{\mathrm{d}\xi}\right|_{max}}{\left|\dfrac{\mathrm{d}f}{\mathrm{d}\xi}\right|_{max}}\frac{1}{\bar{a}_{0min}}\frac{f_{max}}{\bar{a}_{0min}} \ll 1.$$

Therefore, the zero-order approximation $F_0(\xi,\tau)$ given by (A 5) is a solution for (A 2) with an uncertainty of $(f_{max}^2/\bar{a}_{0min}^2)|\mathrm{d}\bar{a}_0(\xi)/\mathrm{d}\xi|_{max}/\mathrm{d}f/\mathrm{d}\xi|_{max} \ll f_{max}$. This result can be presented as

$$\bar{u}(\xi,\tau) = f(\varphi(\xi,\tau)-\bar{u}\tau+O(z)), \tag{A 11}$$

where $z=(f_{max}^2/\bar{a}_{0min}^2)|\mathrm{d}\bar{a}_0/\mathrm{d}\xi|_{max}/\mathrm{d}f/\mathrm{d}\xi|_{max}$.

Appendix B. Asymptotic solution for the shock-wave velocity

At $\tau>\tau_1$, system (4.3a,b) simplifies to:

$$\dot{\xi}_d(\tau) = \bar{a}_0(\xi_{\mathrm{d}}) + \tfrac{1}{2}f_2, \quad \int_{f_2\tau+\xi_L}^{\xi_{\mathrm{d}}(\tau)}\frac{\mathrm{d}x}{\bar{a}_0(x)} = \tau, \tag{B 1a,b}$$

where, $f_2 = f(z_2)$. Differentiating (B 1*b*) and combining the resulting expression for $\dot{\xi}_d(\tau)$ with (B 1*a*), we obtain an equation for $w(\tau) \equiv \tau f_2(\tau)$:

$$\frac{\dot{w}(\tau)\bar{a}_0(\xi_d(\tau))}{\bar{a}_0(w(\tau))} = \frac{1}{2}\frac{w(\tau)}{\tau}.$$

Or, switching from τ to ξ_d:

$$\frac{\mathrm{d}w}{\bar{a}_0(w)w} = \frac{1}{2}\frac{\mathrm{d}\xi_d}{\bar{a}_0(\xi_d)\tau(\xi_d)\dot{\xi}_d(\tau(\xi_d))}. \tag{B 2}$$

For a uniform gas, (i.e. $\bar{a}_0(\xi_d) = 1$) (B 2) has a single solution $w_u = \mathrm{const}\sqrt{\tau}$, which leads to $f(z_2) = w_u/\tau = \mathrm{const}/\sqrt{\tau}$. This, essentially, is the expression obtained earlier by Whitham (1974). Here,

$$\mathrm{const} = \left[2\int_0^{\xi_L} f(y)\,\mathrm{d}y\right]^{1/2}.$$

The assumption of weak shock waves, fundamental for this analysis, allows for integration of (B 2) to a precision of linear terms of $w(\xi_d)/\tau\bar{a}_0(\xi_d)$. Expanding the right-hand side of (B 2) into a Taylor series over this variable and using approximations

$$\dot{\xi}_p(\tau(\xi_d)) = \bar{a}_0(\xi_d) + O\left(\frac{w}{\tau\bar{a}_0(\xi_d)}\right),$$

$$\tau(\xi_d) = \int \frac{\mathrm{d}\xi_d}{\bar{a}_0(\xi_d)} + C + O\left(\frac{w}{\tau\bar{a}_0(\xi_d)}\right),$$

we obtain:

$$\int \frac{\mathrm{d}w}{\bar{a}_0(w)w} = \frac{1}{2}\int \frac{\mathrm{d}\xi_d}{\bar{a}_0^2(\xi_d)\left[\int \dfrac{\mathrm{d}\xi_d}{\bar{a}_0(\xi_d)} + C\right]} + C_1,$$

where C and C_1 are the integration constants which are found from the initial conditions $\xi_d(\tau)|_{\tau=\tau_0} = \xi_{d_0}$, $w(\tau)|_{\tau=\tau_0} = w_0$. Finally, for $\bar{w} = w/w_0$ and $\bar{\xi}_d(\tau) = \xi_d(\tau)/\xi_{d_0}$ we obtain:

$$\int_1^{\bar{w}} \frac{\mathrm{d}y}{\bar{a}_0(y)y} = \frac{1}{2}\int_1^{\bar{\xi}_d} \frac{\mathrm{d}y}{\bar{a}_0^2(y)\left[\dfrac{\tau_0}{\xi_{d_0}} + \displaystyle\int_1^y \frac{\mathrm{d}x}{a_0(x)}\right]}. \tag{B 3}$$

With the same precision, the shock-wave propagation velocity is given by

$$\dot{\bar{\xi}}_d = \bar{a}_0(\xi_d) + \frac{1}{2}\frac{w_0\bar{w}(\xi_d)}{\tau_0\left[1 + \dfrac{\xi_{d_0}}{\tau_0}\displaystyle\int_1^{\bar{\xi}_{d_0}} \frac{\mathrm{d}y}{\bar{a}_0(y)}\right]}.$$

In most of the previous shock-wave propagation experiments through weakly ionized gases, the shock-wave initially travelled through a region with uniform distribution of temperature and density. If $a_0(\xi)$ is constant for $\xi < \xi_{d_0}$ (i.e. $\bar{a}_0(\xi) = 1$), the discontinuity velocity and the Mach number for the region $\bar{\xi}_d \geqslant 1$ can be expressed as:

$$\dot{\bar{\xi}}_d = \bar{a}_0(\xi_d) + \frac{(M_0 - 1)\bar{w}(\xi_d)}{\left[1 + \displaystyle\int_1^{\bar{\xi}_p} \frac{\mathrm{d}y}{\bar{a}_0(y)}\right]}, \quad \frac{M(\bar{\xi}_d) - 1}{M_0 - 1} = \frac{\bar{w}(\xi_d)}{\left(1 + \displaystyle\int_1^{\bar{\xi}_p} \frac{\mathrm{d}y}{\bar{a}_0(y)}\right)\bar{a}_0(\xi_d)}. \tag{B 4}$$

Here M_0 is the Mach number for $\tau = \tau_0$. Using the above, (B 3) transforms into:

$$\int_1^{\bar{w}(\xi_d)} \frac{\mathrm{d}y}{\bar{a}_0(y)y} = \frac{1}{2}\int_1^{\bar{\xi}_d} \frac{\mathrm{d}y}{\bar{a}_0^2(y)\left[1+\displaystyle\int_1^y \frac{\mathrm{d}x}{a_0(x)}\right]}. \tag{B 5}$$

Equation (B 5) can be solved analytically for a non-uniformity restricted in space and when $\tau \gg 1$. If $\bar{a}_0(\xi) = 1$ for $\xi \leqslant \xi_{d_0}$, then $\bar{a}_0(w(\xi_d)) = 1$, since the shock wave is weak, and $\xi_d \gg w(\xi_d)$. This will be valid until $w(\xi_d)$ reaches ξ_{d_0}. It will happen in a period of time of $\Delta\tau \sim \xi_{d_0}/w(\xi_d) \gg 1$. Therefore, for the thermal gradient restricted in space, when the upstream boundary of the thermal gradient is far enough from the coordinate of the shock-wave formation ($\xi_{d_0} \gg 1$), in the left-hand side of (B 5), under the integral, $\bar{a}_0(y) = 1$. Therefore,

$$\bar{w}(\bar{\xi}) = \exp\left\{\frac{1}{2}\int_1^{\bar{\xi}} \frac{\mathrm{d}y}{\bar{a}_0^2(y)\left[1+\displaystyle\int_1^y \frac{\mathrm{d}x}{\bar{a}_0(x)}\right]}\right\}, \tag{B 6}$$

and for the shock-wave Mach number,

$$\frac{M(\bar{\xi})-1}{M_0-1} = \frac{\bar{w}(\bar{\xi})}{\bar{a}_0(\bar{\xi})\left[1+\displaystyle\int_1^{\bar{\xi}} \frac{\mathrm{d}x}{\bar{a}_0(x)}\right]}, \tag{B 7}$$

and shock-wave velocity expressed in real dimensions is,

$$V(\bar{\xi})=a_0(1)\left\{\bar{a}_0(\bar{\xi})+(M_0-1)\exp\left\{\frac{1}{2}\int_1^{\bar{\xi}}\frac{\mathrm{d}y}{\bar{a}_0^2(y)\left[1+\displaystyle\int_1^y \frac{\mathrm{d}x}{\bar{a}_0(x)}\right]}\right\}\Bigg/\left[1+\int_1^{\bar{\xi}} \frac{\mathrm{d}x}{\bar{a}_0(x)}\right]\right\}. \tag{B 8}$$

Here, $\bar{\xi} = \xi_d/\xi_{d_0}$. M_0 and $a_0(1)$ are the shock-wave Mach number and the speed of sound just upstream of the thermal gradient zone, in actual dimensions. This solution can be made more accurate by accounting in (B 1) and (B 2) for linear terms of the series over argument $w(\xi_d)/\tau a_0(\xi_d)$. In such a case, using the expression for the velocity at the discontinuity $\dot{\xi}_P(\tau(\xi_d)) = \bar{a}_0(\xi_d) + \bar{w}(M_0-1)/\bar{\tau}(\xi_d)$ (where $\bar{w}$ is determined by (B 6)) and $\bar{\tau} = \tau/\tau_0$, we obtain:

$$\bar{w}_1(\bar{\xi}) = \exp\left\{\frac{1}{2}\int_1^{\bar{\xi}} \frac{\mathrm{d}y}{\bar{a}_0(y)\left(\bar{a}_0(y)+(M_0-1)\dfrac{\bar{w}(y)}{1+\displaystyle\int_1^y \frac{\mathrm{d}x}{\bar{a}_0(x)}}\right)\left[1+\displaystyle\int_1^y \frac{\mathrm{d}x}{\bar{a}_0(x)}\right]}\right\}, \tag{B 9}$$

$$\left(\frac{M(\bar{\xi})-1}{M_0-1}\right)_1 = \frac{\bar{w}_1(\bar{\xi})}{\bar{a}_0(\bar{\xi})\left[1+\int_1^{\bar{\xi}} \frac{\mathrm{d}x}{\bar{a}_0(x)}\right]}. \tag{B 10}$$

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